Construction of Entanglement-Assisted Quantum Error-Correcting Codes from Classical Linear Codes with Various Hull Dimensions*

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Abstract

We present results on the construction of entanglement-assisted quantum error-correcting codes derived from classical linear codes with various hull dimensions. The parameters of these quantum codes depend not only on the length, dimension, and minimum distance of the underlying classical codes, but also on the dimension of their hulls.

1 Introduction

Let $\mathbb{F}_q^n$ be the n-dimensional vector space over a finite field $\mathbb{F}_q$ with q elements and let it be equipped with an inner product. The (Hamming) *weight* $\mathrm{wt}(x)$ of a vector $x \in \mathbb{F}_q^n$ is the number of its nonzero coordinates. A linear $[n, k, d]$ q-ary code C is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$, and d is the smallest weight among all non-zero codewords of C, called the *minimum weight* (or minimum distance) of the code.

Suppose that $\mathbb{F}_q^n$ is equipped with an inner product. The orthogonal complement of the linear code C with respect to the defined inner product is

$$C^\perp = \{u \in \mathbb{F}_q^n : u \cdot v = 0 \text{ for all } v \in C\}$$

and it is called the *dual (or orthogonal) code* of C. In this work, we use only the standard Euclidean inner product $x \cdot y = \sum_{i=1}^n x_i y_i$ for any two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$. Clearly, $C^\perp$ is a linear $[n, n-k]$ code. The intersection $C \cap C^\perp$ is called the hull of the code and denoted by $\mathcal{H}(C)$. The dimension $h(C)$ of the hull can be at least 0 and at most $\min\{k, n-k\}$. If $h(C) = k$, the code is called self-orthogonal and then $C \subseteq C^\perp$. If $h(C) = 0$, the code is called complementary dual (LCD) code.

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Hulls of linear codes have been of interest due to their rich algebraic structure and various applications [6, 7, 9]. One of the most useful properties of the hull is the construction of entanglement-assisted quantum error-correcting codes (EAQECCs), which are also intensively studied [1]. An EAQECC with parameters $[[n, k, d; c]]$ encodes k logical qubits into n physical qubits with the help of c pre-shared entangled pairs. A relation between the parameters of a linear q-ary code with a given hull dimension and the parameters of an EAQECC is provided in [2]. Specifically, if C is an $[n, k, d]_q$ linear code with dual distance $d^{\perp}$ and hull dimension h, then there exist EAQECCs with parameters $[[n, k-h, d; n-k-h]]_q$ and $[[n, n-k-h, d^{\perp}; k-h]]_q$. This construction is valid not only in the Euclidian case, but also using the Hermitian inner product.

2 Computational results

This research is a continuation of a recent enumeration of optimal linear codes with various hull dimensions [10]. We use the previous computational results to obtain all possible parameters of corresponding EAQECCs. Our idea is to explore these parameters for optimalities in quantum coding theory. Analogously to classical coding theory some bounds on the parameters of EAQECCs have been proved [4, 5, 8].

Our results concern the EA Singleton, EA Hamming, and EA Plotkin bound. Unfortunately, only a few EAQECCs saturate these bounds. As the length n and the entanglement c grow faster than the minimum distance d of the codes, whereas the dimension k and the dimension of the hull h are fixed, the bounds cannot be reached.

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